Math 53, Discussion 107

Wednesday, April 22, 2020 9:31 AM

16.6 (parametric surfaces and their areas)
16.7 (surface integrals)

$x^2+y^2+z^2=R^2$

Example: the sphere of Radius R.

$r(u,v) = R\langle \sin(u)\cos(v), \sin(u)\sin(v), \cos(u)\rangle$

$0 \le u \le \pi \qquad 0 \le v < 2\pi$

$r_u = \frac{\partial r}{\partial u} = R\langle \cos(u)\cos(v), \cos(u)\sin(v), -\sin(u)\rangle$

$r_v = \frac{\partial r}{\partial v} = R\langle -\sin(u)\sin(v), \sin(u)\cos(v), 0\rangle$

if $u=0 \quad v=0$

$r_u = R\langle 1,0,0\rangle$

$r_v = R\langle 0,0,0\rangle$

We can parametrize a surface as $r(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

The tangent plane of a surface at $r(u_0,v_0)$ is:

$$\underbrace{\left(r_u\times r_v\big|_{u_0,v_0}\right)}_{\text{normal}}\cdot\Big(\underbrace{\vec{X}}_{\langle x,y,z\rangle} - \underbrace{r(u_0,v_0)}_{\text{Point}}\Big) = 0$$

The surface area of our surface is written as:

$$A(S) = \iint_{\underset{\text{domain of } r}{D}} \underbrace{\|r_u\times r_v\|}_{dS}\, \underset{du\,dv}{dA} = \iint_S 1\, dS$$

We can integrate a scalar function $f(x,y,z)$ over a surface S:

$$\iint_S f(x,y,z)\,dS = \iint_D f(r(u,v))|r_u \times r_v|\,dA$$

$\boxed{dS} = \|r_u\times r_v\|\,dA$

= infinitesimal area of surface

$\int f\,dx = \sum(\text{value of } f)\underbrace{(\text{width of rectangle})}_{dx}$

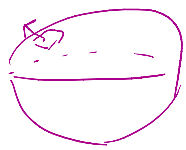
$\boxed{d\vec{S}} = \|r_u \times r_v\|\, dA$
= infinitesimal area of surface

$F: \mathbb{R}^3 \to \mathbb{R}^3$

We can integrate a vector field over a surface by:

$$\underbrace{\iint_S F \cdot d\vec{S}} = \iint_S F \cdot \vec{n}\, dS$$

$\|d\vec{S}\| = dS \quad \|\vec{n}\| = dS$



$\boxed{d\vec{S}} \equiv \vec{n}\, dS$

$= \underbrace{\frac{\vec{r}_u \times \vec{r}_v}{\|\cancel{r_u \times r_v}\|}}_{\vec{n}} \|\cancel{r_u \times r_v}\|\, dA = \underbrace{(\vec{r}_u \times \vec{r}_v)}\, dA$

$\iint_D (F \cdot (\vec{r}_u \times \vec{r}_v))\, (\vec{r}(u,v))$

Exercises

$r(1,1)$ $\quad ax+by+cz+d=0$

1)Find the equation of the tangent plane of the surface at the given point: $x = u^2, y = v^2, z = uv, u = 1, v = 1$

$-(2,2,4)\cdot((x,y,z)-(1,1,1))=0$

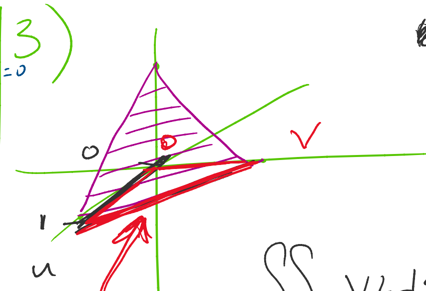
2) Find the area of the surface $z = \frac{2}{3}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}}\right)$ for $0 \le x \le 1, 0 \le y \le 1$

3) Compute $\iint_S yz\, dS$ with S the plane $x + y + z = 1$ in the first octant

$z = 1-x-y$

4) Compute $\int_S F \cdot dS$ for $F = \langle y, x, z^2 \rangle$ and S the heliocoid $\langle u \cos v, u \sin v, v \rangle$ $0 \le u \le 1, 0 \le v \le \pi$

3)



$1-u-v=0$

$v = 1-u$

$r(u,v) = (u, v, \widehat{1-u-v})$

$r_u = (1,0,-1) \quad r_v = (0,1,-1)$

$r_u \times r_v = (1,1,1)$

$\|r_u \times r_v\| = \sqrt{1+1+1} = \sqrt{3}$

$\iint_S yz\, dS = \iint_D v(1-u-v)\, \sqrt{3}\, du\, dv$

$= \sqrt{3} \iint v(1-u-v)\, du\, dv$

$0 \le u \le 1$

$0 \le v \le \underline{1-u}$

4)